Phase-Space White Noise Analysis Application in One-Dimensional Erosion

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Introduction

- White noise analysis provides a natural framework to generalize the notion of infinite - dimensional integrals as one does in the theory of generalized functions or "distributions" [1].
- Phase-space white noise analysis was first developed by Bock and Grothaus [2].
- The random nature of the sand deposition and erosion due to the effects of waves and wind on the seashore can be modeled by the phase-space white noise analysis.
- In the present note we apply phase space white noise analysis to model a one-dimensional erosion in its simplest form.
- Motivated by the Global Soil Erosion data [5, 6] (Data), we calculate for the generalized expectation of the integrand expressed as a white noise functional.

The Model

- The continuous process of erosion allows for the time element to be taken as an independent variable.
- The random nature of the grain/particle deposition with respect to the one-dimensional coordinate space x can be taken as the white noise $\omega_x(\tau)$ random variable,
$$x(\tau) = c_x(t - t_0)\omega_x(\tau), \qquad (1)$$
where $t_0 < \tau < t$ and c_x as constants for proper dimensional units.
- The rate of soil loss by water erosion can be modeled by the Brownian motion $B_p(t)$ that accounts for the fluctuation from certain mean value p_0,
$$p(\tau) = p_0 + c_p \frac{1}{t - t_0} B_p(\tau), \qquad (2)$$
where c_p takes certain physical constants (i.e., to give the entire expression as having units of, e.g., tonnes per hectare per year).
- The ansatz Feynman integrand in phase space with respect to the Gaussian measure μ as proposed in Ref. [4],

$$I_{V,mom} = \mathrm{N} \exp\Bigg(\frac{i}{\hbar} \int_{t_0}^{t} \Big[-x(\tau)\dot{p}(\tau) - \frac{p(\tau)^2}{2m} + \frac{1}{2}\big(\omega_x(\tau)^2 + \omega_p(\tau)^2\big) - V(x(\tau), p(\tau), \tau) \Big] d\tau \Bigg) \cdot \delta(p(t) - p'), \qquad (3)$$

with m being the particle "mass", $\hbar$, the reduced Planck's constant, N, a normalization (in the sequel, we set $\hbar = m = 1$), was motivated by the Heisenberg Uncertainty Principle, that, if the erosion rate is being ascertained (hence with initial value p_0 and with the Brownian fluctuation in momentum, B_p), but the position or extent of erosion x cannot (and thus modeled as not having the initial value but only with white noise, ω_x).

- In our case, we consider the velocity dependent potential,
$$V(x) = -\epsilon D(\tau)\dot{x}. \qquad (4)$$

where $\epsilon \in \mathbb{R}$ is some constant parameter, and $D(\tau)$ is certain time-dependent factor dependent on physical characteristics of the soil that should react to the non - dependent force, which produces such potential.

Example 1 $D(\tau) = 0 \Rightarrow V(x) = 0$. *This is the case where there is no significant "forces" affecting the erosion process.*

Example 2 $D(\tau) = b\tau$, *can be taken as implying a seasonal or temporal dependent erosion. Here the time can have units of months or years.*

The Erosion "Propagator"

- The generalized expectation of the integrand gives the propagator of the model (with primed variables as final values):

$$E(I_{V,mom}) = \delta(p' - p_0) \exp\left(-\frac{i p_0^2 t'}{2}\right) \times \exp\left(i\epsilon D(t') x'\right), \times \exp\left[\frac{\epsilon^2 t \dot{D}^2}{6} - i p_0 t^2 \epsilon \dot{D} + \frac{p_0 t^4 (\epsilon \dot{D})^2}{6}\right]. \qquad (5)$$

Using Data for the Model

- For $D = 0$,
$$E^2 = \exp\left(0.25 p_0^4 t'^2\right). \qquad (6)$$

From the Data, one can calculate E^2, for example, for "Australia", which has the value of $p_0 = 0.75$ with time (in years) $t' = 11$. Certain constants have to be incorporated into the expression Eq. (6) so as to obtain meaningful values.

For example, integrating over the momentum space yields a "probability":
$$P_{p,\delta p} = \int_{p}^{p+\delta p} |E(I_{V,mom})(p)|^2 dp. \qquad (7)$$

- For $D = \tau$, thus, $\dot{D} = 1$
$$E^2 = \exp\left(0.25 p_0^4 t'^2\right) \times \exp\left(t^2 x'^2\right), \times \exp\left[0.028\epsilon^4 t^2 + p_0^2 t^4 \epsilon^2 + 0.028 p_0^2 t^8 \epsilon^2\right], \qquad (8)$$

with which the extent of erosion x' can be included in the model. This value can be calculated from the Data with the assumption of uniform area of erosion.

Outlook

Owing to the random nature at certain level of erosion, phase space white noise analysis can be utilized to model one-dimensional erosion, beginning in simplest forms. This may serve as a springboard to have more sophisticated models on natural phenomena with the aid of computer simulations.

References

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