Phase-Space White Noise Analysis of the Quantum Particle in an Exponentially-Growing Potential

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Introduction

- Feynman integrals are undoubtedly of central importance in varied fields of quantum physics.
- White noise analysis provides a natural framework to generalize the notion of such (infinite - dimensional) integrals as one does in the theory of generalized functions or "distributions" [1].
- In coordinate space, the class of exponentially-growing potentials has been studied in the framework of white noise analysis by Kuna, et. al. [2].
- Phase-space white noise analysis was first developed by Bock and Grothaus [3, 4].
- In the present note we extend in phase space the case of quantum particle with potentials that are Laplace transforms of rapidly decreasing measures such as e.g. the Morse potential.

The Feynman Integrand of the System

- The ansatz Feynman integrand in phase space with respect to the Gaussian measure μ as proposed in Ref. [4],

$$I_{V,mom} = \mathrm{N}\exp\left(\frac{i}{\hbar}\int_{t_0}^{t}\Big[-x(\tau)\dot{p}(\tau) - \frac{p(\tau)^2}{2m}\right) + \frac{1}{2}\Big(\omega_x(\tau)^2+\omega_p(\tau)^2\Big) - V(x(\tau),p(\tau),\tau)\Big]d\tau\Bigg)\cdot\delta(p(t)-p'), \quad (1)$$

with m being the particle mass, $\hbar$, the reduced Planck's constant, N, a normalization, and,

$$x(\tau) = (t-t_0)\omega_x(\tau), \quad (2)$$

$$p(\tau) = p_0 + \frac{1}{t-t_0}B_p(\tau), \quad t_0 < \tau < t; \quad (3)$$

was motivated by the Heisenberg Uncertainty Principle, that, while the momentum as a function of time τ is ascertained (hence with initial value p_0 and with the Brownian fluctuation in momentum, B_p), but the position x is not (and thus modeled as not having the initial value but only with white noise, ω_x).

- In our case, the potential V is given by [2],

$$V(x) = \int_{\mathbb{R}^d} e^{\alpha\cdot x}dm(\alpha). \quad (4)$$

where $m(\alpha)$ is any complex measure with

$$\int_{\mathbb{R}^d} e^{C|\alpha|}d|m|(\alpha) < \infty, \qquad \forall C > 0. \quad (5)$$

Example 1 $V(x) = ge^{ax}$. *Likewise, one obtains potentials such as e.g.* $\sinh(ax)$, $\cosh(ax)$, *and the Morse potential*

$$V(x) = g\left(e^{-2ax} - 2\gamma e^{-ax}\right)$$

with g real, $a, x \in \mathbb{R}^d$ and $\gamma > 0$.

Example 2 *A Gaussian measure m gives the anharmonic oscillator potentials $V(x) = ge^{bx^2}$. Entire functions of arbitrary high order of growth are also in this class.*

- Remarkably, in each case the construction of the Feynman integrand is *perturbative*.
- We formally expand the exponential into a perturbation series w.r. to V. This leads to

$$I_{V,mom} = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t}d^ns\int_{\mathbb{R}^{dn}} I_{0,mom} \times\exp\left(\sum_{l=1}^{n}\alpha_l\cdot x(s_l)\right)\prod_{l=1}^{n}dm(\alpha_l), \quad (6)$$

where $I_{0,mom}$ is the momentum-space free particle Feynman integrand given by,

$$I_{0,mom} = \mathrm{N}\exp\left(\frac{i}{\hbar}\int_{t_0}^{t}\Big[-x(\tau)\dot{p}(\tau) - \frac{p(\tau)^2}{2m}\right) + \frac{1}{2}\Big(\omega_x(\tau)^2+\omega_p(\tau)^2\Big)\Big]d\tau\Bigg) \times\delta(p(t)-p'). \quad (7)$$

- We show that the rhs of Eq.6 defines a generalized function of white noise using the characterization theorem, the corollary on Bochner integrability, and the corollary on convergence in the space of Kondratiev distributions from Refs. [1,5-7]. We do this in three steps: for the integrand, for the integral, and finally for the sum.

The Quantum Propagator of the System

- The generalized expectation of the Feynman integrand gives the quantum propagator of the system, and is given by (with x' being the final position and p' being the final momentum),

$$E(I_{V,mom}) = \delta(p'-p_0)\exp\left(-\frac{ip_0^2t}{2}\right)\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t}d^ns\int_{\mathbb{R}^{dn}}\prod_{l=1}^{n}dm(\alpha_l) \times\exp\left(\sum_{l=1}^{n}\alpha_l\cdot x'\right), \times\exp\Bigg[\frac{it}{6}\sum_j\sum_k\alpha_j\alpha_k(t-s_j) + p_0t\sum_l\alpha_l(t-s_l) - \frac{ip_0t^2}{6}\sum_j\sum_k\alpha_j\alpha_k(t-s_j)(t-s_k)\Bigg], \quad (8)$$

which also solves the corresponding Schrödinger equation.

Conclusion

The phase space white noise analysis proves useful in calculating for the probability of finding a quantum particle in an exponentially-growing potential moving a certain momentum that is conserved, with an additional information on the final position of the particle as expressed in Eq. 8.

References

[1] T. Hida, H. H. Kuo, J. Potthoff and L. Streit, *White Noise. An Infinite Dimensional Calculus* (Kluwer Academic, 1993).
[2] T. Kuna, L. Streit and W. Westerkamp, *J. Math. Phys.* **39**, 4476 (1998).
[3] W. Bock and M. Grothaus, *Theor. Probability and Math. Statist.* **85**, 722 (2012).
[4] W. Bock, *Hamiltonian Path Integrals in White Noise Analysis* (PhD dissertation, Department of Mathematics, University of Kaiserlautern, 2013).
[5] Z.-Y. Huang and J.-A. Yan, *Introduction to Infinite Dimensional Stochastic Analysis* (Kluwer, 2000).
[6] H. H. Kuo, *White Noise Distribution Theory* (CRC Press, 1996).
[7] N. Obata, *White Noise Calculus and Fock Space* (Springer LNM 1577, 1994).